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PATHOS AND TOTAL PATHOS SEMIFULL BLOCK GRAPH

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Abstract

In this paper, we introduce the concept of pathos semifull block graph and total pathos semifull block graph of a tree. We obtain some properties of these graphs. We study the characterization of graphs whose pathos and total pathos semifull block graph are planar, maximal outer planar, minimally nonouter planar, crossing number one.

1. Introduction

All graphs considered here are finite, undirected without loops or multiple edges. Any undefined term or notation in this paper may be found in Harary [2].

The concept of pathos of a graph G was introduced by [1] as a collection of minimum number of edge disjoints open paths whose union is G. The path number of a graph G is the number of path of pathos. Stanton [9] and Harary [2] have calculated the path number of certain classes of graphs like trees and complete graphs.

Key Words : *Inner vertex number, Pathos graph, Semifull line and Semifull block graph.*
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For a graph $G(p,q)$ if $B = \{u_1, u_2, u_3, \cdots, u_r; r \geq 2\}$ is a block of G, then we say that point u_1 and block B are incident with each other, as are u_2 and B and so on. If two distinct blocks B_1 and B_2 are incident with a common cut vertex then they are called adjacent blocks.

The crossing number $c(G)$ of G is the least number of intersection of pairs of edges in any embedding of G in the plane. Obviously G is planar if and only if $c(G) = 0$. The *edgedegree* of an edge $e = \{a, b\}$ is the sum of degrees of the end vertices a and b. Degree of a block is the number of vertices lies on a block. *Blockdegree* B_v of a vertex v is the number of blocks in which v lies. *Blockpath* is a path in which each edge in a path becomes a block. If two paths p_1 and p_2 contain a common cutvertex then they are adjacent paths and the *pathdegree* P_v of a vertex v is the number of paths in which v lies. Pendant pathos is a path p_i of pathos having unit length.

The inner vertex number $i(G)$ of a planar graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane.

A new concept of a graph valued functions called the edge semientire block graph $E_b(G)$ of a plane graph G was introduced by Venkanagouda in [11] and is defined as the graph whose vertex set is the union of set of edges, set of blocks and set of regions of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent, corresponding blocks are adjacent, the corresponding edges lies on the blocks, the corresponding edges lies on the region.

The pathos edge semientire graph $P_e(T)$ of a tree was introduced by in [12]. The pathos edge semientire graph $P_e(T)$ of a tree T is the graph whose vertex set is the union of set of edges, regions and the set of pathos of pathos in which two vertices are adjacent if and only if the corresponding edges of T are adjacent, edges lies on the region and edges lies on the path of pathos. Since the system of path of pathos for a tree T is not unique, the corresponding pathos edge semientire graph is also not unique.

The pathos edge semientire block graph of a tree T denoted by $P_{vb}(T)$ is the graph whose vertex set is the union of the vertices, regions, blocks and path of pathos of T in which two vertices are adjacent if and only if they are adjacent vertices of T or vertices lie on the blocks of T or vertices lie on the regions of T or the adjacent blocks of T. Clearly the number of regions in a tree is one. This concept was introduced by

Venkanagouda in [3].

The block graph $B(G)$ of a graph G is the graph whose vertex set is the set of blocks of G in which two vertices are adjacent if the corresponding blocks are adjacent. This graph was studied in [2]. The path graph $P(T)$ of a tree is the graph whose vertex set is the set of path of pathos of T in which two vertices of $P(T)$ are adjacent if the corresponding path of pathos have a common vertex.

The semifull graph $F_s(G)$ of a graph G is the graph whose vertex set is the union of vertices, edges and blocks of G in which two vertices are adjacent if the corresponding members of G are adjacent or one corresponds to a vertex and other to a line incident with it or one corresponds to a block B of G and other to a vertex v of G and v is in B. This concept was introduced in [5]. Further in [6] they studied the concept of semifull block graph $F_b(G)$ and is defined as the graph whose vertex set is the union of vertices, edges and blocks of G in which two vertices are adjacent if the corresponding vertices and blocks of G are adjacent or one corresponds to a vertex and other to an edge incident with it or one corresponds to a block B of G and other to a vertex v of G and v is in B. The pathos line graph of a tree is introduced in [7] and is defined as the graph whose vertex set is the union of the set of edges and paths of pathos of T in which two vertices are adjacent if and only if the corresponding edges of T are adjacent and the edge lies on the path of pathos of T.

The following will be useful in the proof of our results.

Theorem A [6] : If G is a (p, q) connected graph, whose vertices have degree d_i and if b_i is the number of blocks to which vertex v_i belongs in G, then the semifull block graph $F_b(G)$ of G has $q + \Sigma b_i + 1$ vertices and $3q + \frac{1}{2}\Sigma b_i(b_i + 1)$ edges.

Theorem B [7] : The pathos line graph $P_l(T)$ of a tree T is planar if and only if $\Delta(T) \leq 4$.

Theorem C [1] : A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Theorem D [1] : Every maximal outerplanar graph G with p vertices has $(2p - 3)$ edges.

Theorem E [1] : A connected graph G is eulerian if and only if each vertex in G has even degree.

Theorem F [2] : A nontrivial graph is bipartite if and only if all its cycles are even.

Theorem G [2] : If G is a (p, q) graph whose vertices have degree d_i then $L(G)$ has q vertices and q_L edges where $q_L = -q + \frac{1}{2}\Sigma d_i^2$.

Theorem H [2] : The line graph $L(G)$ of a graph G has crossing number one if and only if G is planar and 1 or 2 holds:

1. The maximum degree $d(G)$ is 4 and there is unique non cutvertex of degree.

2. The maximum degree $d(G)$ is 5, every vertex of degree 4 is a cutvertex and there is a unique vertex of degree 5 and has at most 3 edges in any block.

2. Pathos Semifull Block Ggraph

We now define the following graph valued function.

Definition 2.1 : The pathos semifull block graph $P_{Fb}(T)$ of a tree T is the graph whose vertex set is the union of the set of vertices, edges, blocks and path of pathos of T in which two vertices of $P_{Fb}(T)$ are adjacent if the corresponding vertices are adjacent and corresponding blocks are adjacent or one corresponds to a vertex of T and other to an edge incident with it or one corresponds to a block b of T and other to a vertex v of T and v is in b or one corresponds to edge e of T and other to a path of pathos piof T and e is on p_i. A tree T and the corresponding pathos semifull block graph $P_{Fb}(T)$ of a tree T are shown in the figure 2.1.

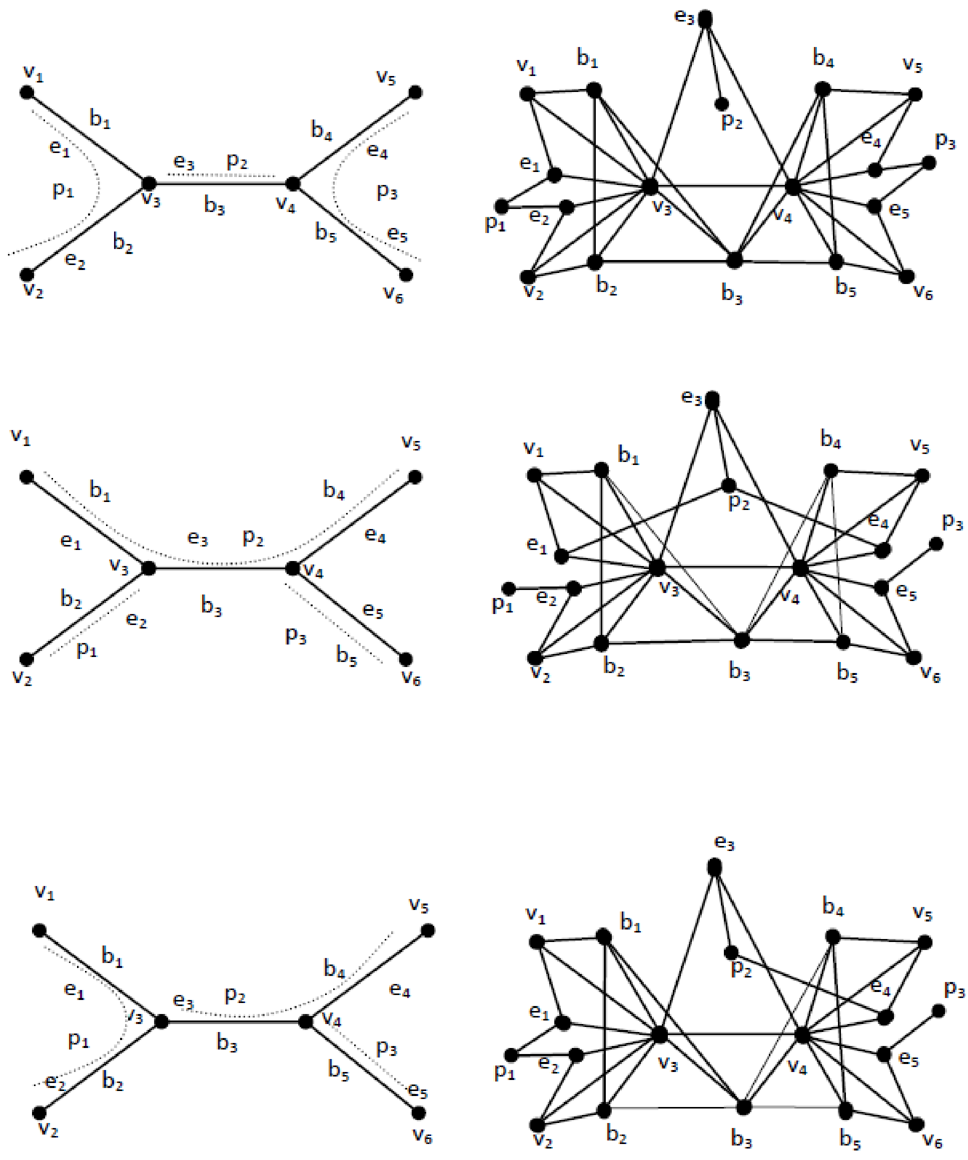
We start with some remarks.

Remark 1 : If a tree T is connected then $P_{Fb}(T)$ is also connected and conversely.

Remark 2 : The semifull block graph $F_b(T)$ of a tree T is a spanning sub graph of $P_{Fb}(T)$.

Remark 3 : For every edge e_i of a tree T, the corresponding vertex e'_i in $P_{Fb}(T)$ having degree three.

Remark 4 : If the degree of vertex v_i in a tree T is n, then the degree of the corresponding vertex v'_i in $P_{Fb}(T)$ is $3n$.



T **Figure 2.1** $\mathbf{P_{Fb}(T)}$

3. Main Results

Theorem 1 : For any (p, q) tree T, $P_{Fb}(T)$ is nonseparable if T does not contains pendant pathos.

Proof : Consider T be a tree. Let $e_1, e_2, \cdots, e_{p-1}$ be the edges, $b_1 = e_1$, $b_2 = e_2, \cdots, b_{p-1} = e_{p-1}$ be the blocks, $v_1, v_2, \cdots, v_p$ be the vertices and $p_1, p_2, \cdots, p_k$ be the path of pathos of T. By definition of semifull block graph, $F_b(T)$ is always nonseparable. If T contains a pendant pathos p_i with an edge e_k then in $P_{Fb}(T)$, a corresponding vertex p'_i is adjacent to e'_k which forms a cut vertex, a contradiction hence T does not contains pendent pathos.

Theorem 2 : If T is tree without isolated vertices then $P_{Fb}(T)$ is not a bipartite graph.

Proof : Let T be a tree without isolated vertex, then T has a block $b_i = e_i$. Let $v_i, v_{i+1} \in b_i$, since b_i is incident with the vertices v_i and v_{i+1}, it implies that the corresponding vertices b_i, v_i, v_{i+1} form a cycle C_3 in $P_{Fb}(T)$. By Theorem D, $P_{Fb}(T)$ is not a bipartite graph.

Theorem 3 : If T be a (p, q) connected tree whose vertices have degree d_i and if b_i is the number of blocks to which vertices v_i belongs in T, then the pathos semifull block graph $P_{Fb}(T)$ of T has $q + \Sigma b_i + k + 1$ vertices and $4q + \frac{1}{2}\Sigma b_i(b_i + 1)$ edges where k be the path number in T.

Proof : By Remark 2, $F_b(T)$ is an spanning sub graph $P_{Fb}(T)$. By the definition of $P_{Fl}(T)$, the number of vertices $P_{Fl}(T)$ is the sum of the number of vertices of $F_b(T)$ and the number of path of pathos. By Theorem A the number of vertices of $F_b(T)$ is $q + \Sigma b_i + 1$ and the number of path of pathos is k. Hence $P_{Fb}(T)$ has $q + \Sigma b_i + 1 + k$ vertices.

Further by Remark 2, the number of edges in $P_{Fb}(T)$ is the number of edges in $F_b(T)$ and the number of edges lies in each path of pathos. By Theorem A, $F_b(T)$ has $3q + \frac{1}{2}\Sigma b_i(b_i + 1)$ edges and number of edges lies in each path of pathos is q. Hence the number of edges in

$$\begin{aligned} P_{Fb}(T) &= 3q + \frac{1}{2}\Sigma b_i(b_i + 1) + q \\ &= 4q + \frac{1}{2}\Sigma b_i(b_i + 1). \end{aligned}$$

Theorem 4 : The pathos semifull block graph $P_{Fb}(T)$ of a tree T is planar if and only if $\Delta(T) \leq 3$ for every vertex v of T.

Proof : Suppose $P_{Fb}(T)$ is planar. Assume that $\Delta(T) \leq 4$. If there exist a vertex v of degree 4 in T, then in $F_{Fb}(T)$, this vertex v together with the block vertices form K_5 as an induced subgraph. Further the corresponding pathosvertices are adjacent to the vertices e'_i which are the edges lies on path of pathos p_i, which gives $P_{Fb}(T)$ nonplanar, a contradiction.

Conversely, Suppose $\Delta(T) \leq 3$ for every vertex v of T. Clearly $F_b(T)$ is planar. Let $[b_1, b_2, b_3, \cdots, b_{p-1}]$ be the blocks of T with p vertices such that $b_1 = e_1$, $b_2 = e_2, \cdots, b_{p-1} = e_{p-1}$, P_i be the number of pathos of T now V $P_{Fb}(T) = V(G) \cup \{b'_1, b'_2, \cdots, b'_{p-1}\} \cup \{P_i\}$ where $\{b'_1, b'_2, \cdots, b'_{p-1}\}$ be the blockvertices in $P_{Fb}(T)$ corresponds to the block $b_1, b_2, \cdots, b_{p-1}$ of T. By definition of $P_{Fb}(T)$, the embedding of $P_{Fb}(T)$ in any plane gives a planar graph.

Theorem 5 : The pathos semifull block graph $P_{Fb}(T)$ of a tree T is minimally nonouter planar if and only if $T = P_3$.

Proof : Suppose $P_{Fb}(T)$ is minimally nonouterplanar, assume $T = P_4 = \{e_1, e_2, e_3\}$. By the definition of $F_b(T)$, $F_b(T)$ each edge of T form K_{4-x} in $F_b(T)$. Clearly $F_b(T)$ is outer planar. In $P_{Fb}(T)$, the pathos vertex p_i is adjacent to all the edges e_1, e_2, e_3 which givesthe inner vertex number $i[P_{Fb}(T)] > 1$, a contradiction.

Conversely suppose $T = P_3(e_1e_2)$. By definition of $F_b(T)$, $F_b(P_3)$ is outerplanar. In $P_{Fb}(T)$, the pathos vertex p_1 is adjacent to vertices e'_1, e'_2 which corresponds the edges e_1, e_2 to form a graph with v_2 as inner vertex. Hence $i[P_{Fb}(P_3)] = 1$.

Theorem 6 : For any tree T, $P_{Fb}(T)$ is always noneulerian.

Proof : Suppose T be any tree. By Remark 3 degree of the vertex e'_i of $P_{Fb}(T)$ corresponds to the edge e_i of T is always 3. By Theorem C $P_{Fb}(T)$ is noneulerian.

Theorem 7 : For any tree T, $P_{Fb}(T)$ is Hamiltonian if and only if T has no pendant pathos.

Proof : Suppose T has no pendant pathos we have the following cases.

Case 1 : Let T be a path $P_n : v_1e_1v_2e_2 \cdots e_{P-1}v_p$ and $b_1 = e_1$, $b_2 = e_2 \cdots b_{p-1} = e_{p-1}$ be the blocks of T then it has exactly one path of pathos p_1. The vertex set of $T_{Fb}(T)$ is

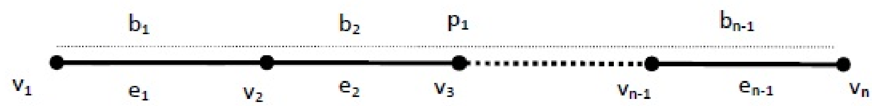
$$V[P_{Fb}(T)] = V(T) \cup \{e'_1, e'_2, \cdots, e'_{p-1}\} \cup \{b'_1, b'_2, \cdots, b'_{p-1}\} \cup p_i.$$

In $P_{Fb}(T)$, the given graph T as a sub graph. The block vertex b_i is adjacent to b_{i+1} and v_i is adjacent to e_i $\forall$ i as shown in Figure 2.2. Clearly in $P_{Fb}(T)$, the Hamiltonian cycle $p_1e_1v_1b_1\ v_2e_2v_3\ b_2\cdots b_{n-1}v_pe_{p-1}P_1$ exists. Hence $P_{Fb}(T)$ is Hamiltonian graph.

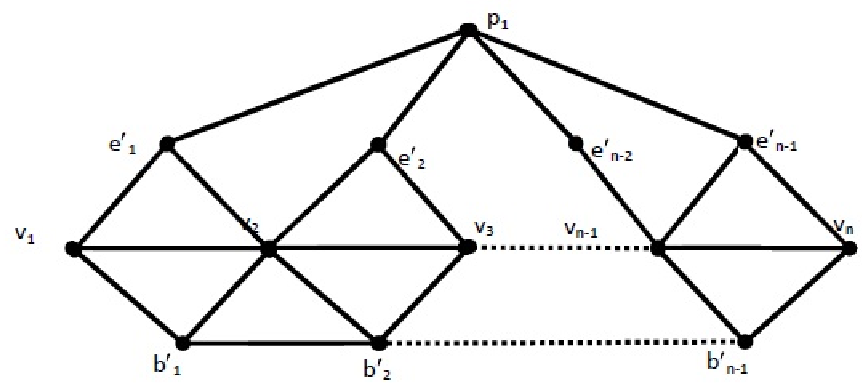
Case 2 : Suppose T is not a path, Then T has at least one vertex with degree at least three. Let

$$V[P_{Fb}(T)] = V(T) \cup \{e'_1, e'_2, \cdots, e'_{p-1}\} \cup \{b'_1, b'_2, \cdots, b'_{p-1}\} \cup \{p_1, p_2, \cdots, p_k\}.$$

Then in $P_{Fb}(T)$, there exist a cycle contain all the vertices of $P_{Fb}(T)$ has $p_1e_1v_1b_1v_3e_3p_2e_4\cdots b_{p-1}v_pe_{p-1}p_1$. Hence $P_{Fb}(T)$ is Hamiltonian graph.
Conversely part is obvious.



P_n



$P_{Fb}(T)$

Figure 2.2

4. Total Pathos Semifull Block Graph $T_{Fb}(T)$

Now we define another graph valued function.

The total pathos semifull block graph $T_{Fb}(T)$ of a tree T is the graph whose vertex set is the union of the set of vertices, edges, blocks and path of pathos of T in which two vertices $T_{Fb}(T)$ are adjacent if the corresponding vertices are adjacent, or blocks are adjacent or paths of pathos are adjacent or one corresponds to a vertex of T and other to an edge incident with it or one corresponds to a Block b of T and other to a vertex v of T and v is in b or one corresponds to an edge e of T and other to a path of pathos p_i of T and e is on p_i. A tree T and the corresponding $T_{Fb}(T)$ are shown in the Figure 2.3.

Remark 5 : For any tree T, the pathos semifull block graph $P_{Fb}(T)$ is a spanning sub graph of $T_{Fb}(T)$.

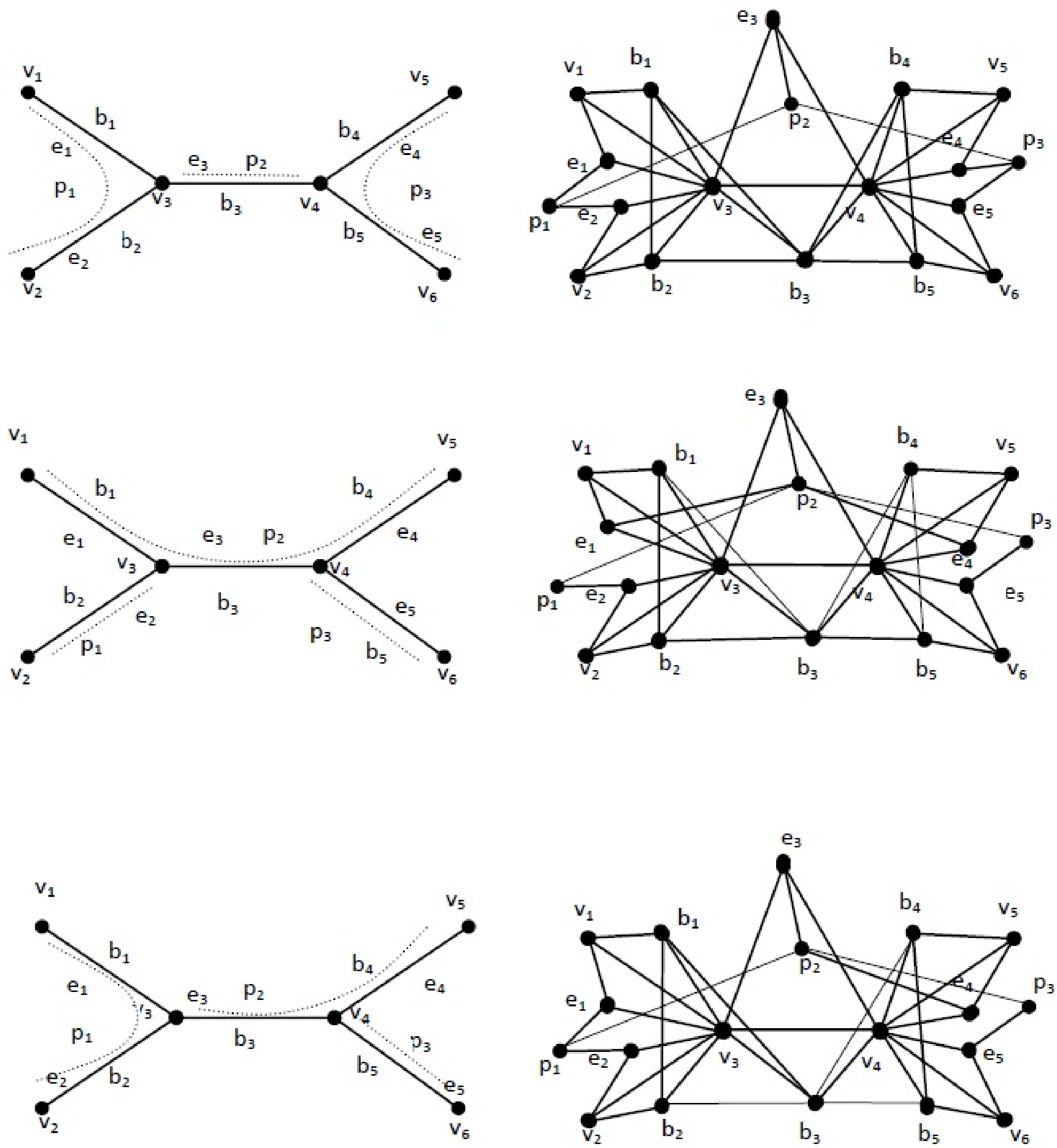
Theorem 8 : For any tree T, the total pathos semi full block graph $T_{Fb}(T)$ is always nonseparable.

Proof: Consider a tree T. Let $e_1, e_2, \cdots, e_{p-1}$ be the edges, $b_1 = e_1, b_2 = e_2, \cdots, .b_{p-1} = e_{p-1}$ be the blocks, $v_1, v_2, \cdots, v_p$ be the vertices of T and $p_1, p_2, \cdots, p_k$ be the path of pathos of T. By the definition of semifull block graph $F_b(T)$ is always nonseparable. By Theorem 1, $P_{Fb}(T)$ is nonseparable. Suppose T does not contains a pendant pathos. Since $T_{Fb}(T)$ the pathos vertices are adjacent, and each pathosvertex is adjacent to at least one another pathos vertex, it follows that there is no cut vertex in $T_{Fb}(T)$. Hence $T_{Fb}(T)$ is always nonseparable.

Theorem 9 : If T is a (p, q) connected tree whose vertices have degree d_i and if b_i is the number of blocks to which vertices v_i belongs in T, then the total pathos semifull block graph $T_{Fb}(T)$ of T has $q + \Sigma b_i + k + 1$ vertices and $4q + \frac{1}{2}\Sigma b_i(b_i + 1) + \frac{1}{2}\Sigma p_{d_i}(p_{d_i} - 1)$ edges where p_{d_i} be the path degree of vertex v_i and k be the path number in T.

Proof : By Remark 5, $P_{Fb}(T)$ is a spanning subgraph of $T_{Fb}(T)$. The number of vertices of $T_{Fb}(T)$ is equal to the number of vertices of $P_{Fb}(T)$. By the Theorem 3, the number of vertices of $P_{Fb}(T)$ is $q + \Sigma b_i + k + 1$ vertices. Hence the number of vertices of $T_{Fb}(T) = q + \Sigma b_i + k + 1$.

Further by the definition of $T_{Fb}(T)$, the number of edges of $T_{Fb}(T)$ is the sum of number of edges of $P_{Fb}(T)$ and the number obtain by the path degree of a vertex p_{d_i}, which is $\frac{1}{2}\Sigma p_{d_i}(p_{d_i} - 1)$. By theorem 3 the number of edges in $P_{Fb}(T)$ is $4q + \frac{1}{2}\Sigma b_i(b_i + 1)$. Hence the number of edges in $T_{Fb}(T) = 4q + \frac{1}{2}\Sigma b_i(b_i+1) + \frac{1}{2}\Sigma p_{d_i}(p_{d_i}-1)$.



T **Figure 2.3** **$T_{Fb}(T)$**

Theorem 10 : For any tree T, $T_{Fb}(T)$ is always nonplanar.

Proof : By Theorem 4, $P_{Fb}(T)$ is planar if $\Delta(T) \leq 3$ for every vertex v of T. Suppose $T = k_{1,3}$, T contains two path of pathos p_1 and p_2. By the definition $P_{Fb}(T)$, $P_{Fb}(K_{1,3})$ is planar and both the pathosvertices lies different regions. In $T_{Fb}(T)$ joining the vertices p_1 and p_2 gives nonplanar. Hence $T_{Fb}(T)$ is always nonplanar.

Theorem 11 : For any tree T, $T_{Fb}(T)$ is always noneulerian.

Proof : Suppose T be any tree. By Remark 3 degree of a vertex e'_i in $P_{Fb}(T)$ corresponds to the edge e_i of T is always 3. In $T_{Fb}(T)$ degree e'_i is same as in $P_{Fb}(T)$. By Theorem C $T_{Fb}(T)$ is noneulerian.

Theorem 12 : For any tree T, $T_{Fb}(T)$ is always Hamiltonian.

Proof : By Theorem 7, $P_{Fb}(T)$ is Hamiltonian if T has no pendent pathos and hence $T_{Fb}(T)$ is Hamiltonian. Suppose T has pendant pathos. In $P_{Fb}(T)$ there exist a Hamiltonian path. Since in $T_{Fb}(T)$, the pathos vertices are adjacent, it implies that there exists a Hamiltonian cycle in $T_{Fb}(T)$. Hence $T_{Fb}(T)$ is Hamiltonian.

4. Conclusion

In this paper, we introduced the concept of the pathos semifull block graph and total pathos semifull block graph of a tree. We characterized the graphs whose pathos semifull block graph and total pathos semifullblock graph are planar, noneulerian, Hamiltonian and crossing number one.

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